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Improving the Asymptotics for the Greatest Zeros of Hermite Polynomials*

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Abstract—The Liouville-Stekloff method for approximating solutions of homogeneous linear ODE and a general result due to Tricomi which provides estimates for the zeros of functions by means of the knowledge of an asymptotic representation are used in order to improve a classical asymptotic formula for the greatest zero of the n^{th} Hermite polynomial.

Keywords—Asymptotic expansions, Hermite polynomials, Zeros, Volterra integral equations, Liouville-Stekloff method.

1. INTRODUCTION

Let $H_n(\xi)$ be the n^{th} Hermite polynomial which belongs to the Orthogonal Polynomial Set in $(-\infty, \infty)$, with respect to the weight function $e^{-\xi^2}$.

Denote by $\xi_{1,n} > \xi_{2,n} > \cdots > \xi_{n,n}$ the zeros of $H_n(\xi)$, enumerated in decreasing order, and by $i_1 < i_2 < i_3 < \cdots$ the real zeros of the Airy's function $\mathcal{A}(x)$ ($i_1 \cong 3.37213\ldots$).

REMARK I. The Airy's function we consider here is a solution of the ODE $y'' + 1/3xy = 0$ (see [1, pp. 18,19]). In the sequel, we mention also the more usual standardization, by considering the solutions $\mathcal{A}i(x)$, $\mathcal{B}i(x)$ of the ODE $y'' - xy = 0$ (see [2, pp. 253–256]).

It is well known (see [1, p. 132]) that the asymptotic behavior of the greatest zero, when $n \to \infty$, is $\xi_{1,n} \simeq (2n+1)^{1/2}$. More precisely, the following asymptotic formula is proved

$$\frac{\xi_{1,n}}{(2n+1)^{1/2}} = 1 - \frac{i_1}{6^{1/3}(2n+1)^{2/3}} + o(n^{-2/3}). \tag{1.1}$$

General results connected with the above problem are given by A. Maté, P. Nevai and V. Totik in [3].

It is shown below that it is possible to obtain more terms in expansion (1.1) by using the following mathematical tools:

- a method described by F. G. Tricomi (see [4]), which allows us to deduce estimates for the zeros of a function f by means of knowledge of an asymptotic representation of f;

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- the Liouville-Stekloff method, by means of which asymptotic expansions of the solutions of an homogeneous linear ODE are obtained in terms of the known solutions of a basic equation.

After a short describing of the above-mentioned tools, given in the Sections 2 and 3, the required application to the asymptotics for the greatest zero of the Hermite polynomial $H_n(\xi)$ is obtained (see Sections 4 and 5).

2. A RESULT DUE TO F. G. TRICOMI

In the article [4], F. G. Tricomi proved the following result:

PROPOSITION I. *Suppose the continuous function $f(x)$ admits (uniformly with respect to x), the asymptotic representation*

$$f(x) = \sum_{k=0}^{m} g_k(x)\mu^k + \mathcal{O}(\mu^{m+1}), \quad \mu \to 0, \tag{2.1}$$

where the functions $g_k(x)$ are differentiable $m-k+1$ times in a neighborhood of a point x_0 which is a simple zero of the function $g_0(x)$ ($g_0(x_0) = 0$, $g_0'(x_0) \neq 0$). Suppose further that $g_m(x) \in C^1$ in the same neighborhood. Then $\forall \varepsilon > 0$ and for $|\mu|$ less than a suitable $\delta > 0$, the equation $f = 0$ is satisfied at least by a value x_0^* s.t. $|x_0^* - x_0| < \varepsilon$, and the following expansion holds

$$x_0^* = x_0 + \sum_{k=1}^{m} \omega_{k-1}\mu^k + \mathcal{O}(\mu^{m+1}), \tag{2.2}$$

where the coefficients $\omega_0, \omega_1, \omega_2, \dots$ are rational functions of the values

$$G_{k,\ell} := \frac{1}{\ell!} g_k^{(\ell)}(x_0) \ ,$$

and are determined by the system

$$\begin{aligned}
0 &= G_{10} + G_{01}\omega_0, \\
0 &= G_{20} + G_{01}\omega_1 + G_{11}\omega_0 + G_{02}\omega_0^2, \\
0 &= G_{30} + G_{01}\omega_2 + G_{11}\omega_1 + 2G_{02}\omega_0\omega_1 + G_{21}\omega_0 + G_{12}\omega_0^2 + G_{03}\omega_0^3, \\
&\vdots
\end{aligned}$$

(see [4], for the general equation).

The first expressions for the ω_k are given by

$$\omega_0 = -\frac{G_{10}}{G_{01}} = -\frac{g_1(x_0)}{g_0'(x_0)}, \tag{2.3}$$

$$\begin{aligned}
\omega_1 &= -\frac{G_{10}^2 G_{02} - G_{10}G_{01}G_{11} + G_{01}^2 G_{20}}{G_{01}^3} \\
&= -\frac{\frac{1}{2}g_1^2(x_0)g_0''(x_0) - g_1(x_0)g_0'(x_0)g_1'(x_0) + \left(g_0'(x_0)\right)^2 g_2(x_0)}{\left(g_0'(x_0)\right)^3}.
\end{aligned} \tag{2.4}$$

Note that formula (2.2) provides an asymptotic estimate for a zero x_0^* of a function f in terms of the zero x_0 of g_0, provided that the representation (2.1) is known.

3. THE GENERAL FORM OF THE LIOUVILLE-STEKLOFF METHOD

In a book of F. G. Tricomi (see [5]) can be found a method, ascribed to G. Fubini, which can be interpreted as a general form of the Liouville-Stekloff method.

Tricomi considers a second order homogeneous linear ODE

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0, \tag{3.1}$$

with $P_0(x) \neq 0$ in $(a, b) \subseteq \mathbb{R}$.

Suppose that, putting

$$P_0(x) = 1 - \alpha(x), \qquad P_1(x) = p_1(x) - \beta(x), \qquad P_2(x) = p_2(x) - \gamma(x), \tag{3.2}$$

and writing (3.1) in the form

$$y'' + p_1(x)y' + p_2(x)y = \alpha(x)y'' + \beta(x)y' + \gamma(x)y, \tag{3.3}$$

the following conditions are satisfied.

I. The approximate ODE

$$y'' + p_1(x)y' + p_2(x)y = 0, \tag{3.4}$$

can be integrated explicitly. To one or both the linearly independent integrals $F_1(x)$, $F_2(x)$ could be imposed, in some case, other restrictions (e.g., stability, i.e., boundedness when x increases).

II. The remainder functions $\alpha(x)$, $\beta(x)$, $\gamma(x)$ satisfy, with respect to a parameter λ, and for a suitable $r > 0$, the conditions

$$\alpha(x) = \mathcal{O}\left(\lambda^{-r}\right), \quad \beta(x) = \mathcal{O}\left(\lambda^{-r}\right), \quad \gamma(x) = \mathcal{O}\left(\lambda^{-r}\right). \tag{3.5}$$

Denote by

$$W(x) := \begin{vmatrix} F_1(x) & F_2(x) \\ F_1'(x) & F_2'(x) \end{vmatrix} \tag{3.6}$$

the Wronskian of the above integrals and put

$$\mathcal{G}_h(x) := \frac{\alpha(x)F_h''(x) + \beta(x)F_h'(x) + \gamma(x)F_h(x)}{[1-\alpha(x)]W(x)}, \qquad h = 1, 2, \tag{3.7}$$

$$L(x,\xi) := \begin{vmatrix} F_1(\xi) & F_2(\xi) \\ F_1(x) & F_2(x) \end{vmatrix}; \tag{3.8}$$

then the proposition below is true.

PROPOSITION II. *Anyone of two independent solutions $Y_1(x)$, $Y_2(x)$ of the original equation* (3.1) *admits the representation*

$$Y_h(x) = F_h(x) + \int_{a_0}^{x} L(x,\xi)\mathcal{G}_h(\xi)\,d\xi + \sum_{s=1}^{\infty}\int_{a_0}^{x} L(x,\xi)\left[\int_{a_0}^{\xi} K_s(\xi,\eta)\mathcal{G}_h(\eta)\,d\eta\right] d\xi, \tag{3.9}$$

where $a_0 \in (a,b)$, $h = 1,2$, and the kernels $K_s(x,\eta)$ are defined by the induction formulas

$$K_1(\xi,\eta) := K(\xi,\eta) := \begin{vmatrix} F_1(\eta) & F_2(\eta) \\ \mathcal{G}_1(\xi) & \mathcal{G}_2(\xi) \end{vmatrix}, \tag{3.10}$$

$$K_s(\xi,\eta) = \int_{\eta}^{\xi} K(\xi,z)K_{s-1}(z,\eta)\,dz, \qquad s > 1. \tag{3.11}$$

By Proposition II, and taking into account condition (3.5) and the consequent relations

$$\mathcal{G}_h(x) = \mathcal{O}\left(\lambda^{-r}\right), \qquad h = 1,2, \tag{3.12}$$

$$K_1(\xi,\eta) = \mathcal{O}\left(\lambda^{-r}\right), \quad K_2(\xi,\eta) = \mathcal{O}\left(\lambda^{-2r}\right), \dots, K_s(\xi,\eta) = \mathcal{O}\left(\lambda^{-sr}\right), \dots \tag{3.13}$$

the approximation formulas, of increasing precision, for the integrals of equation (3.1) immediately follow

$$Y_h(x) = F_h(x) + \mathcal{O}\left(\lambda^{-r}\right), \tag{3.14}$$

$$Y_h(x) = F_h(x) + \int_{a_0}^{x} L(x,\xi)\mathcal{G}_h(\xi)\,d(\xi) + \mathcal{O}\left(\lambda^{-2r}\right), \tag{3.15}$$

$$Y_h(x) = F_h(x) + \int_{a_0}^{x} L(x,\xi)\mathcal{G}_h(\xi)\,d\xi + \int_{a_0}^{x} L(x,\xi)\left[\int_{a_0}^{\xi} K_1(\xi,\eta)\mathcal{G}_h(\eta)\,d\eta\right] d\xi + \mathcal{O}\left(\lambda^{-3r}\right), \tag{3.16}$$

($h = 1,2$), and so on. That is, *formula* (3.9), *under the hypothesis* (3.5), *provides asymptotic estimates with respect to parameter λ (as $\lambda \to \infty$) for the integrals of the original equation* (3.1), *of any prescribed level of accuracy.*

4. APPLICATION OF THE GENERALIZED LIOUVILLE-STEKLOFF METHOD TO THE HERMITE CASE

It is well known (see [1, p. 106]) that putting

$$z(\xi) = e^{-\frac{\xi^2}{2}} H_n(\xi), \tag{4.1}$$

$$h_n := (2n+1)^{1/2}, \quad \xi = h_n - x, \tag{4.2}$$

$$y(x) = z\big(\xi(x)\big) = \exp\left(-\frac{(h_n - x)^2}{2}\right) H_n(h_n - x), \tag{4.3}$$

the function $y(x)$ satisfies the ODE

$$y'' + 2h_n xy = x^2 y. \tag{4.4}$$

By writing (4.4) in the form

$$\frac{1}{h_n} y'' + 2xy = \frac{1}{h_n} x^2 y, \tag{4.5}$$

it is possible to apply the Liouville-Stekloff method to the approximating equation

$$\frac{1}{h_n} y'' + 2xy = 0, \tag{4.6}$$

which is equivalent to the following

$$y'' + 2h_n xy = 0. \tag{4.7}$$

The last equation is satisfied by the Airy's functions

$$F_1(x) = \mathcal{A}\big((6h_n)^{1/3} x\big), \quad F_2(x) = \mathcal{B}\big((6h_n)^{1/3} x\big).$$

REMARK II. Note that, in the present case, the coefficients, and consequently the solutions of the "approximate" differential equation (4.7) *are depending on n*. Nevertheless, the methods exposed in the preceding Sections 2 and 3 remain true. The only difference will be the necessity of *a more careful evaluation of the infinitesimal orders in the asymptotic estimates.*

By recalling that $W\big(\mathcal{A}i(x), \mathcal{B}i(x)\big) = \frac{1}{\pi}$ (see e.g., [2, p. 254]), and the relations between two different standardizations

$$\mathcal{A}(x) = \frac{\pi}{3^{1/3}} \mathcal{A}i\left(-\frac{x}{3^{1/3}}\right);$$

$$\mathcal{B}(x) = \frac{\pi}{3^{1/3}} \mathcal{B}i\left(-\frac{x}{3^{1/3}}\right),$$

we can write

$$W\big(F_1(x), F_2(x)\big) = -\frac{\pi}{3}(6h_n)^{1/3}.$$

Furthermore,

$$L(x,\xi) := \mathcal{A}\big((6h_n)^{1/3}\xi\big)\,\mathcal{B}\big((6h_n)^{1/3}x\big) - \mathcal{A}\big((6h_n)^{1/3}x\big)\,\mathcal{B}\big((6h_n)^{1/3}\xi\big). \tag{4.8}$$

By comparing (4.4) with (3.3), we deduce

$$\alpha(x) = \beta(x) \equiv 0; \qquad \gamma(x) = \frac{1}{h_n} x^2 = \mathcal{O}(n^{-1/2}). \tag{4.9}$$

Recalling an asymptotic property for the Airy's functions (see [2, p. 256]), we can write

$$\mathcal{G}_1(x) = -\frac{3x^2 \mathcal{A}\big((6h_n)^{1/3}x\big)}{\pi h_n (6h_n)^{1/3}} = \mathcal{O}(n^{-17/24}). \tag{4.10}$$

$$\mathcal{G}_2(x) = -\frac{3x^2 \mathcal{B}\big((6h_n)^{1/3}x\big)}{\pi h_n (6h_n)^{1/3}} = \mathcal{O}(n^{-17/24}). \tag{4.11}$$

As a consequence of Proposition I, we have the following theorem.

THEOREM I. *Using preceding hypotheses and notations, a solution $Y_1(x)$ of the ODE (4.4) admits the following representation*

$$\begin{aligned} Y_1(x) = {} & \mathcal{A}\big((6h_n)^{1/3}x\big) - \frac{3}{\pi h_n (6h_n)^{1/3}} \int_0^x \xi^2 L(x,\xi)\mathcal{A}\big((6h_n)^{1/3}\xi\big) d\xi \\ & + \sum_{s=1}^{\infty} \left[\frac{-3}{\pi h_n (6h_n)^{1/3}}\right]^{s+1} \int_0^x \xi^2 L(x,\xi) \left[\int_0^\xi \eta^2 L_s(\xi,\eta)\mathcal{A}\big((6h_n)^{1/3}\eta\big) d\eta\right] d\xi, \end{aligned} \tag{4.12}$$

where the iterated kernels $L_s(\xi,\eta)$ are defined by the induction formulas

$$\begin{aligned} L_1(\xi,\eta) &:= L(\xi,\eta), \\ L_s(\xi,\eta) &:= \int_\eta^\xi z^2 L(\xi,z) L_{s-1}(z,\eta)\, dz, \qquad s > 1. \end{aligned} \tag{4.13}$$

PROOF. It is sufficient to write the representation defined by formulas (3.9)–(3.11), remarking that, in this case, we have

$$K_1(\xi,\eta) = \begin{vmatrix} \mathcal{A}\big((6h_n)^{1/3}\eta\big) & \mathcal{B}\big((6h_n)^{1/3}\eta\big) \\ \dfrac{-3\xi^2}{\pi h_n (6h_n)^{1/3}}\mathcal{A}\big((6h_n)^{1/3}\xi\big) & \dfrac{-3\xi^2}{\pi h_n (6h_n)^{1/3}}\mathcal{B}\big((6h_n)^{1/3}\xi\big) \end{vmatrix} = \frac{-3\xi^2}{\pi h_n (6h_n)^{1/3}} L(\xi,\eta),$$

and, by induction

$$K_s(\xi,\eta) = \left[\frac{-3}{\pi h_n (6h_n)^{1/3}}\right]^s \xi^2 L_s(\xi,\eta).$$

We can write, indeed

$$\begin{aligned} K_s(\xi,\eta) &= \int_\eta^\xi K(\xi,z) K_{s-1}(z,\eta)\, dz \\ &= \frac{-3}{\pi h_n (6h_n)^{1/3}} \xi^2 \int_\eta^\xi L(\xi,z) K_{s-1}(z,\eta)\, dz \\ &= \left[\frac{-3}{\pi h_n (6h_n)^{1/3}}\right]^s \xi^2 \int_\eta^\xi z^2 L(\xi,z) L_{s-1}(z,\eta)\, dz \\ &= \left[\frac{-3}{\pi h_n (6h_n)^{1/3}}\right]^s \xi^2 L_s(\xi,\eta). \end{aligned}$$

Then, for $s = 1$

$$\begin{aligned} & \int_0^x L(x,\xi) \left[\int_0^\xi K_1(\xi,\eta)\mathcal{G}_1(\eta)\, d\eta\right] d\xi \\ & \quad = \int_0^x L(x,\xi) \left[\int_0^\xi \frac{3\xi^2}{\pi h_n (6h_n)^{1/3}} L(\xi,\eta) \frac{3\eta^2}{\pi h_n (6h_n)^{1/3}} \mathcal{A}\big((6h_n)^{1/3}\eta\big)\, d\eta\right] d\xi \\ & \quad = \left[\frac{3}{\pi h_n (6h_n)^{1/3}}\right]^2 \int_0^x \xi^2 L(x,\xi) \left[\int_0^\xi \eta^2 L_1(\xi,\eta)\mathcal{A}\big((6h_n)^{1/3}\eta\big)\, d\eta\right] d\xi; \end{aligned}$$

and, in general

$$\int_0^x L(x,\xi)\left[\int_0^\xi K_s(\xi,\eta)\mathcal{G}_1(\eta)\,d\eta\right]d\xi$$

$$= \int_0^x L(x,\xi)\left[\int_0^\xi\left[\int_\eta^\xi K(\xi,z)K_{s-1}(z,\eta)\,dz\right]\mathcal{G}_1(\eta)\,d\eta\right]d\xi$$

$$= \left[\frac{-3}{\pi h_n(6h_n)^{1/3}}\right]^s \int_0^x \xi^2 L(x,\xi)\left[\int_0^\xi\left[\int_\eta^\xi z^2 L(\xi,z)L_{s-1}(z,\eta)\,dz\right]\mathcal{G}_1(\eta)\,d\eta\right]d\xi$$

$$= \left[\frac{-3}{\pi h_n(6h_n)^{1/3}}\right]^s \int_0^x \xi^2 L(x,\xi)\left[\int_0^\xi L_s(\xi,\eta)\mathcal{G}_1(\eta)\,d\eta\right]d\xi$$

$$= \left[\frac{-3}{\pi h_n(6h_n)^{1/3}}\right]^{s+1} \int_0^x \xi^2 L(x,\xi)\left[\int_0^\xi \eta^2 L_s(\xi,\eta)\mathcal{A}\big((6h_n)^{1/3}\eta\big)\,d\eta\right]d\xi.$$

By Theorem I, we deduce, in particular, the following asymptotic formulas for the solution $Y_1(x)$ of the ODE equation (4.4)

$$Y_1(x) = \mathcal{A}\big((6h_n)^{1/3}x\big) + \mathcal{O}\left(n^{-19/24}\right), \tag{4.14}$$

$$Y_1(x) = \mathcal{A}\big((6h_n)^{1/3}x\big) - \frac{3}{\pi h_n(6h_n)^{1/3}}\int_0^x \xi^2 L(x,\xi)\mathcal{A}\big((6h_n)^{1/3}\xi\big)\,d\xi + \mathcal{O}\left(n^{-37/24}\right), \tag{4.15}$$

$$\vdots$$

$$\begin{aligned} Y_1(x) = {} & \mathcal{A}\big((6h_n)^{1/3}x\big) - \frac{3}{\pi h_n(6h_n)^{1/3}}\int_0^x \xi^2 L(x,\xi)\mathcal{A}\big((6h_n)^{1/3}\xi\big)\,d\xi \\ & + \sum_{s=2}^m \left[\frac{-3}{\pi h_n(6h_n)^{1/3}}\right]^s \int_0^x \xi^2 L(x,\xi)\left[\int_0^\xi \eta^2 L_{s-1}(x,\eta)\mathcal{A}\big((6h_n)^{1/3}\eta\big)\,d\eta\right]d\xi \\ & + \mathcal{O}\left(n^{-(18m+19)/24}\right). \end{aligned} \tag{4.16}$$

REMARK III. The last relations provide representation formulas for the function $Y_1(x)$ of the same kind needed in Proposition I. It is sufficient to assume

$$\mu = \frac{1}{h_n} \simeq n^{-1/2}, \qquad \text{for } n \to \infty.$$

Since the hypotheses of this proposition are clearly satisfied in our case, we can infer asymptotic representations of any order of accuracy for the zeros of $Y_1(x)$ in terms of the zeros of the Airy's function $\mathcal{A}\big((6h_n)^{1/3}x\big)$ and of the function $L(x,\xi)$, by means of which the coefficients of formula (4.16) are expressed.

5. ASYMPTOTIC ESTIMATES FOR THE GREATEST ZERO OF THE HERMITE POLYNOMIAL $H_n(\xi)$

Let

$$g_0(x) := \mathcal{A}\big((6h_n)^{1/3}x\big), \tag{5.1}$$

$$g_1(x) := \frac{-3}{\pi(6h_n)^{1/3}}\int_0^x \xi^2 L(x,\xi)\mathcal{A}\big((6h_n)^{1/3}\xi\big)\,d\xi, \tag{5.2}$$

$$g_2(x) := \left[\frac{-3}{\pi(6h_n)^{1/3}}\right]^2 \int_0^x \xi^2 L(x,\xi)\left[\int_0^\xi \eta^2 L(\xi,\eta)\mathcal{A}\big((6h_n)^{1/3}\eta\big)\,d\eta\right]d\xi, \tag{5.3}$$

$$g_m(x) := \left[\frac{-3}{\pi(6h_n)^{1/3}}\right]^m \int_0^x \xi^2 L(x,\xi)\left[\int_0^\xi \eta^2 L_{m-1}(\xi,\eta)\mathcal{A}\big((6h_n)^{1/3}\eta\big)\,d\eta\right]d\xi, \tag{5.4}$$

and write (4.16) in the form

$$Y_1(x) = g_0(x) + (g_1(x))\frac{1}{h_n} + (g_2(x))\left(\frac{1}{h_n}\right)^2 + \cdots + (g_m(x))\left(\frac{1}{h_n}\right)^m + \mathcal{O}\left(n^{-(18m+19)/24}\right). \quad (5.5)$$

By assuming $\mu = 1/h_n$, in Proposition I, *and remarking that the coefficients $g_k(x)$ in formula* (5.5) *are depending on n*, we deduce the following asymptotic estimates for the greatest zero x_0^* of $Y_1(x)$, which makes use of the nearest zero $x_0 := i_1/(6h_n)^{1/3}$ of approximating function $\mathcal{A}((6h_n)^{1/3}x)$.

– From formula (4.14)

$$x_0^* = \frac{i_1}{(6h_n)^{1/3}} + \mathcal{O}\left(n^{-4/3}\right); \quad (5.6)$$

– From formula (4.15)

$$x_0^* = \frac{i_1}{(6h_n)^{1/3}} + \omega_0\frac{1}{h_n} + \mathcal{O}\left(n^{-5/2}\right), \quad (5.7)$$

where

$$\begin{aligned}\omega_0 &= \frac{3}{\pi(6h_n)^{2/3}}\frac{\mathcal{B}(i_1)}{\mathcal{A}'(i_1)}\int_0^{i_1/(6h_n)^{1/3}} \xi^2\left[\mathcal{A}((6h_n)^{1/3}\xi)\right]^2 d\xi \\ &= \frac{3}{\pi(6h_n)^{5/3}}\frac{\mathcal{B}(i_1)}{\mathcal{A}'(i_1)}\int_0^{i_1} x^2\mathcal{A}^2(x)\,dx;\end{aligned} \quad (5.8)$$

– From formula (4.16) by assuming $m = 2$

$$x_0^* = \frac{i_1}{(6h_n)^{1/3}} + \omega_0\frac{1}{h_n} + \omega_1\left(\frac{1}{h_n}\right)^2 + \mathcal{O}\left(n^{-11/3}\right), \quad (5.9)$$

with ω_0 given by formula (5.8), and

$$\omega_1 = -\frac{g_1^2(x_0)g_0''(x_0) - 2g_1(x_0)g_0'(x_0)g_1'(x_0) + 2\left(g_0'(x_0)\right)^2 g_2(x_0)}{2\left(g_0'(x_0)\right)^3}, \quad (5.10)$$

where

$$g_1(x_0) = -\frac{3\mathcal{B}(i_1)}{\pi(6h_n)^{4/3}}\int_0^{i_1} x^2\mathcal{A}^2(x)\,dx, \quad (5.11)$$

$$g_1'(x_0) = \frac{1}{2\pi h_n}\left[\mathcal{A}'(i_1)\int_0^{i_1} x^2\mathcal{A}(x)\mathcal{B}(x)\,dx - \mathcal{B}'(i_1)\int_0^{i_1} x^2\mathcal{A}^2(x)\,dx\right], \quad (5.12)$$

$$g_0'(x_0) = \mathcal{A}'(i_1)(6h_n)^{1/3}; \quad g_0''(x_0) = \mathcal{A}''(i_1)(6h_n)^{2/3}, \quad (5.13)$$

$$\begin{aligned}g_2(x_0) = \frac{9\mathcal{B}(i_1)}{\pi^2(6h_n)^{8/3}}\Bigg[&\int_0^{i_1} x^2\mathcal{A}(x)\mathcal{B}(x)\int_0^x \xi^2\mathcal{A}^2(\xi)\,d\xi\,dx \\ &- \int_0^{i_1} x^2\mathcal{A}^2(x)\int_0^x \xi^2\mathcal{A}(\xi)\mathcal{B}(\xi)\,d\xi\,dx\Bigg].\end{aligned} \quad (5.14)$$

By returning to the original variable ξ, we can finally proclaim the results below.

THEOREM II. *For the greatest zero of the n^{th} Hermite polynomial, we can write the asymptotic estimates*

$$\xi_{1,n} = h_n - \frac{i_1}{(6h_n)^{1/3}} + \mathcal{O}\left(n^{-4/3}\right), \quad (5.15)$$

$$\xi_{1,n} = h_n - \frac{i_1}{(6h_n)^{1/3}} - \omega_0\frac{1}{h_n} + \mathcal{O}\left(n^{-5/2}\right), \quad (5.16)$$

$$\xi_{1,n} = h_n - \frac{i_1}{(6h_n)^{1/3}} - \omega_0\frac{1}{h_n} - \omega_1\left(\frac{1}{h_n}\right)^2 + \mathcal{O}\left(n^{-11/3}\right). \quad (5.17)$$

By putting $a_1 = -i_1/3^{1/3} \cong -2.33810\dots$, it is easily seen that

$$\omega_0 \frac{1}{h_n} = -\frac{1}{h_n^{8/3}} \frac{\pi}{2^{5/3}} \left[\frac{\mathcal{B}i(a_1)}{\mathcal{A}i'(a_1)} \int_0^{-a_1} x^2 \mathcal{A}i^2(-x)\, dx \right] \cong \frac{0.32823\dots}{(2n+1)^{4/3}}, \tag{5.18}$$

$$\begin{aligned}
\omega_1 \left(\frac{1}{h_n}\right)^2 = \frac{1}{h_n^5} \frac{\pi^2}{2^4} \frac{\mathcal{B}i(a_1)}{[\mathcal{A}i'(a_1)]^3} \Bigg\{ & \mathcal{B}i(a_1)\mathcal{A}i''(a_1) \left[\int_0^{-a_1} x^2 \mathcal{A}i^2(-x)\, dx \right]^2 \\
& + 2\mathcal{A}i'(a_1) \int_0^{-a_1} x^2 \mathcal{A}i^2(-x)\, dx \Bigg[\mathcal{A}i'(a_1) \int_0^{-a_1} x^2 \mathcal{A}i(-x)\mathcal{B}i(-x)\, dx \\
& \qquad - \mathcal{B}i'(a_1) \int_0^{-a_1} x^2 \mathcal{A}i^2(-x)\, dx \Bigg] \\
& - 2[\mathcal{A}i'(a_1)]^2 \Bigg[\int_0^{-a_1} x^2 \mathcal{A}i(-x)\mathcal{B}i(-x) \int_0^{x} y^2 \mathcal{A}i^2(-y)\, dy\, dx \\
& \qquad - \int_0^{-a_1} x^2 \mathcal{A}i^2(-x) \int_0^{x} y^2 \mathcal{A}i(-y)\mathcal{B}i(-y)\, dy\, dx \Bigg] \Bigg\} \\
\cong -\frac{0.0072757\dots}{(2n+1)^{5/2}}. &
\end{aligned} \tag{5.19}$$

Then the preceding formulas (5.15),(5.16),(5.17) are, respectively, equivalent to

$$\frac{\xi_{1,n}}{(2n+1)^{1/2}} = 1 - \frac{i_1}{6^{1/3}(2n+1)^{2/3}} + \mathcal{O}\left(n^{-11/6}\right), \tag{5.20}$$

$$\frac{\xi_{1,n}}{(2n+1)^{1/2}} = 1 - \frac{i_1}{6^{1/3}(2n+1)^{2/3}} - \frac{0.32823\dots}{(2n+1)^{11/6}} + \mathcal{O}\left(n^{-3}\right), \tag{5.21}$$

$$\frac{\xi_{1,n}}{(2n+1)^{1/2}} = 1 - \frac{i_1}{6^{1/3}(2n+1)^{2/3}} - \frac{0.32823\dots}{(2n+1)^{11/6}} + \frac{0.0072757\dots}{(2n+1)^3} + \mathcal{O}\left(n^{-25/6}\right). \tag{5.22}$$

REMARK IV. Note that, even in the first case, formula (5.20) seems to be better than (1.1), since it shows the exact infinitesimal order.

REMARK V. After this paper was finished, the author was informed about results of L. Gatteschi (see [6]) on uniform approximation for the zeros of Laguerre Polynomials. Nevertheless, his results are obtained by using a different approximating differential equation, so that the resulting asymptotics for the zeros, in particular for the greatest zero, are different from those obtained in this article.

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